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Abstract

We analyse an approach aimed at determining statistical properties of spectra
of a time-periodic quantum chaotic system based on the parameter dynamics of
their quasienergies. In particular we show that application of the methods of
statistical physics, proposed previously in the literature, taking into account
appropriate integrals of motion of the parametric dynamics is fully justified,
even if the used integrals of motion do not determine the invariant manifold
in a unique way. The indetermination of the manifold is removed by applying
Dirac's theory of constrained Hamiltonian systems and imposing appropriate
primary, first-class constraints and a gauge transformation generated by them
in the standard way. The obtained results close the gap in the whole reasoning
aiming at understanding statistical properties of spectra in terms of parametric
dynamics.

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1. Introduction

One of the most characteristic features of quantum systems which exhibit chaotic behaviour
in the classical limit is an affinity of their spectral properties to random matrices. The famous
Bohigas–Giannoni–Schmidt conjecture [1] states that the statistics of distances between
neighbouring energy levels of a quantum system with chaotic classical limit is well described
by that derived from the random matrix theory (RMT) [2].

A vast numerical and experimental evidence [3, 4] in favour of this hypothesis was
collected during last 20 years. There are also convincing theoretical arguments supporting it
[5, 6]. In the present paper we would like to reconsider one of the first theoretical approaches

initiated by Pechukas [7] and further developed by Yukawa [8, 9]. The original idea consisted in deriving differential equations describing parametric level dynamics i.e. the evolution of eigenvalues, when the parameter controlling the amount of chaos in the system changes, and applying the rules of classical equilibrium statistical mechanics to the flow described by the derived differential equations, treating the parameter as a fictitious time in which the 'evolution' takes place.

Pechukas started with the quantum Hamiltonian of the form

$$H = H_0 + \lambda V, \tag{1}$$

where, according to the original interpretation, a time-independent $N \times N$ Hermitian matrix H_0 represented a quantum system enjoying integrable classical limit, whereas V was an integrability-breaking part making the whole system classically fully chaotic when λ attained appropriately large values. Writing the Schrödinger equation in the form

$$H|\phi_m\rangle = q_m|\phi_m\rangle, \tag{2}$$

differentiating over the perturbation parameter λ and taking matrix elements in the energy eigenbasis $\{|\phi_n\rangle_{n=1,\dots,N}\}$ one arrives, after appropriate choice of the (otherwise irrelevant) phases of the eigenvectors, at a closed system of differential equations,

$$\frac{\mathrm{d}q_n}{\mathrm{d}\lambda} = p_n, \tag{3}$$

$$\frac{\mathrm{d}p_n}{\mathrm{d}\lambda} = -\sum_{k \neq n} l_{nk}l_{kn}\mathcal{V}'(q_k - q_n), \tag{4}$$

$$\frac{\mathrm{d}l_{mn}}{\mathrm{d}\lambda} = -\sum_{k \neq m,n} l_{mk}l_{kn}(\mathcal{V}(q_n - q_k) - \mathcal{V}(q_k - q_m)), \tag{5}$$

where $p_n = \langle\phi_n|V|\phi_n\rangle$, $l_{mn} = \langle\phi_n|V|\phi_n\rangle(q_m - q_n)$ for $n \neq m$, $l_{nn} = 0$, $\mathcal{V}(q) = -1/q^2$, and $'$ in (4) denotes the derivative with respect to the argument.

As observed by Yukawa, the dynamical system (3)–(5), in which the parameter λ was treated as a fictitious time, is a Hamiltonian one, i.e. the equations can be written in the form

$$\frac{\mathrm{d}q_n}{\mathrm{d}\lambda} = \{\mathcal{H}, q_n\}, \qquad \frac{\mathrm{d}p_n}{\mathrm{d}\lambda} = \{\mathcal{H}, p_n\}, \qquad \frac{\mathrm{d}l_{mn}}{\mathrm{d}\lambda} = \{\mathcal{H}, l_{mn}\}, \tag{6}$$

with the Hamilton function

$$\mathcal{H} = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \frac{1}{2}\sum_{n,m=1}^{N} l_{mn}l_{nm}\mathcal{V}(q_n - q_m), \tag{7}$$

provided the Poisson brackets among the phase-space variables q_n, p_n and l_{mn} are given as

$$\{p_m, q_n\} = \delta_{mn}, \qquad \{p_m, p_n\} = \{q_m, q_n\} = 0, \tag{8}$$

$$\{l_{mn}, l_{ij}\} = \delta_{in}l_{mj} - \delta_{mj}l_{in}, \tag{9}$$

$$\{p_m, l_{kn}\} = \{q_m, l_{kn}\} = 0. \tag{10}$$

Applying rules of equilibrium statistical mechanics was straightforward; the equilibrium distribution should be given as the Boltzmann one,

$$\rho = \mathcal{N}\exp(-\beta\mathcal{H}), \tag{11}$$

where $\mathcal{H}$ is the Hamilton function of the system (7), β—a fictitious temperature (to be determined in some way from the initial data), and $\mathcal{N}$—an appropriate normalization constant.

Integration over the variables p_n and l_{mn} led to the equilibrium distribution of energy levels q_n. As shown by Pechukas and Yukawa the resulting distribution coincides with that provided by RMT for the ensemble of real symmetric matrices with identically and independently distributed elements (forming the so called Gaussian orthogonal ensemble).

As appealing and straightforward as the above outlined approach might be, one should not overlook some fundamental obstacles appearing when attempting to formulate it in a more rigorous way. Let us summarize briefly the most disturbing of them. First, the parametric motion in the energy levels of (1) is clearly unbounded—the eigenvalues of H grow indefinitely and, without an additional scaling, no 'equilibrium' distribution of eigenvalues of H is attained (although it might be that the statistics of distances measured in units of the mean distance approaches some 'equilibrium'). There is a way of curing the situation—one should turn to dynamics of rescaled energy levels, which in fact consists in changing the λ-dependence of H (see [3]). In the present paper we will be interested in the parametric evolution of the eigenphases of the one-period propagator for the time-periodic systems. The eigenphases are from the definition bound to the finite interval form 0 to 2π, so the above-mentioned conceptual (or technical) problem does not arise.

Now the most important remains the following trouble. The postulated Boltzmann distribution can be validated only if there are no other constants of the motion apart from the Hamilton function itself, or in other words, when the motion is ergodic on the whole constant-energy surface. In the case of Pechukas–Yukawa parametric dynamics, as well as in the case of parametric motion of the periodic systems it is not the case. The dynamical systems governing the parametric motion of eigenvalues or eigenphases are so called generalized Calogero–Moser or Sutherland–Moser systems [10]. They possess many additional integrals of motion, and in fact the motion is ergodic on a much smaller invariant manifold [11]. The simplest way to include the influence of additional integrals of motion consists of using in place of canonical ensemble measure (11) its grand-canonical generalization

$$\rho \propto \exp\left\{-\sum_\mu \beta_\mu I_\mu\right\}, \tag{12}$$

nailing down the invariant manifold on which the motion takes place by fixing constants of the motion I_μ in the ensemble mean with the help of Lagrange parameters β_μ; one of these I_μ should be the Hamilton function $\mathcal{H}$. Using the microcanonical ensemble $\rho \propto \prod_\mu \delta(I_\mu - \overline{I}_\mu)$ fixing the values of the constants of motion I_μ to their initial values $\overline{I}_\mu$ exactly, rather in the ensemble mean as (12) does, would be even more appropriate, but technically more complicated, see [3], chapter 6 for a discussion of the problem.

Integration of ρ over all dynamical variables except the eigenphases yields the desired distribution of the latter. Such a program was performed in [12, 13] where it was shown that inclusion of additional known integrals of motion leads to corrections of the order $1/N$ in comparison with the predictions of RMT. This result is highly satisfactory, since one expects convergence to RMT in the limit when the dimension of the matrix N tends to infinity (what in considered models corresponds to the classical limit of the quantum systems). The only remaining problem is whether the integrals of motion taken into account in [12, 13] are all, which are needed to fix (in the ensemble mean) the invariant manifold on which the motion is ergodic, or in other words, what is the minimal set of independent integrals of motion determining the invariant manifold (see also [3] for the formulation of the problem). The independence of the integrals used in the above-mentioned papers was investigated in [14]. In the present paper we close the last gap by showing that they determine the invariant manifold to the extent which is appropriate to infer the distribution of eigenphases.

2. Parametric eigenphases dynamics

We propose to start with the propagator for a particularly simple time-dependent quantum system in which the integrability breaking part in (1) has a form of periodic instantaneous kicks, so the whole Hamiltonian reads now

$$H(\lambda) = H_0 + \lambda V \sum_{n=-\infty}^{\infty} \delta(t - nT), \tag{13}$$

where V is some constant, Hermitian $N \times N$ matrix and T the period of the perturbation.

One of the most prominent examples of such systems is the, so called, kicked top [3, 15, 16] for which H_0 describes a linear precession around one axis, whereas V is a nonuniform rotation around a perpendicular axis with the angular velocity dependent on the angular momentum. In effect, both H_0 and V are polynomials (respectively linear and quadratic) in the angular momentum operators. The model is capable of exhibiting chaos in the classical limit and conforms very accurately to predictions of RMT concerning statistical properties of spectra. It also does not exhibit the phenomenon of the dynamical localization (observed e.g. in such models like the kicked rotator and other kicked systems) which would lead to spectral correlations different from those predicted by RMT (for the explanation of lack of localization see [3], chapter 7.6 and the literature cited therein).

In the case of a kicked system (13), the one-period unitary evolution operator F (the propagator), transporting a state vector of the system over one period T of the perturbation, takes a particulary simple form

$$F(\lambda) = \exp(-i\lambda V) F_0, \qquad F_0 := \exp(-iH_0), \tag{14}$$

where for simplicity we put $T = 1$ and $\hbar = 1$.

Of our interest will be the eigenphases (quasienergies) $q_n(\lambda)$ of $F(\lambda)$,

$$F(\lambda)|\phi_n(\lambda)\rangle = \exp(iq_n(\lambda))|\phi_n(\lambda)\rangle, \tag{15}$$

where $|\phi_n(\lambda)\rangle$ are eigenvectors of $F(\lambda)$.

Let a unitary $N \times N$ matrix $W(\lambda)$ diagonalize $F(\lambda)$,

$$W(\lambda)F(\lambda)W^{-1}(\lambda) = e^{-iQ(\lambda)} = \text{diag}(e^{-iq_1(\lambda)}, \ldots, e^{-iq_N(\lambda)}), \tag{16}$$

$$Q(\lambda) = \text{diag}(q_1(\lambda), \ldots, q_N(\lambda)). \tag{17}$$

In the following we shall skip exhibiting the explicit λ-dependence when possible. In the case of a general unitary matrix, the diagonalizing matrix W is not unique even after ordering the eigenphases: we can always left-multiply it by a diagonal unitary matrix without altering the result (16).

Let us define following auxiliary matrices:

$$v := WVW^{-1} = v^\dagger, \tag{18}$$

$$l := i\,e^{iQ}[v, e^{-iQ}] = -l^\dagger. \tag{19}$$

In the following we denote by v_{mn} and l_{mn}, $m, n = 1, \ldots, N$ the matrix elements of v and l.

Differentiating (16) over λ we arrive at

$$\frac{dQ}{d\lambda} = i(a - e^{-iQ}a\,e^{iQ}) + v, \tag{20}$$

$$\frac{dv}{d\lambda} = [a, v], \tag{21}$$

$$\frac{\mathrm{d}l}{\mathrm{d}\lambda} = [a, l],\tag{22}$$

where

$$a = \frac{\mathrm{d}W}{\mathrm{d}\lambda} W^{-1}.\tag{23}$$

By an appropriate choice of the diagonalizing matrix W, we can choose $a_{nn} = 0$.

In the following we shall use the notation $v_{nn} = p_n$. In terms of individual matrix elements equations (20)–(22) give exactly (3)–(5), despite a different meaning of the variables q_n, p_n and l_{mn} [3, 16, 17]. The only difference is that now

$$\mathcal{V}(q) = -\frac{1}{4\sin^2\frac{q}{2}}.\tag{24}$$

Obviously, the equations again are Hamiltonian with λ treated as a fictitious time, and the Poisson brackets (8)–(10).

The formal analogy between the autonomous and the kicked case is by no means accidental. For a unified treatment of parametric in both cases see [18, 19].

Now the system of equations (3)–(5) can be treated as describing dynamics (in the fictitious time λ) of a one-dimensional gas of particles on the unit circle interacting mutually *via* the potential (24), but with evolving 'coupling strengths' l_{mn} becoming thus additional dynamical variables.

3. Statistical mechanics of the gas of eigenphases

As the phase-space trajectory of the fictitious gas evolves in the fictitious time λ, the original matrix $F(\lambda)$ changes within a one-parameter family. During the evolution 'particles of the gas' (i.e. in fact, the eigenphases) undergo mutual collisions. Observe that since the potential in (7) is repulsive, they usually do not cross (i.e. do not exchange positions). Such a real crossing of two eigenphases is possible only if the respective l_{mn} vanishes. Instead what is usually observed in the region of parameter λ corresponding to classically chaotic behaviour, are so called avoided crossings when two neighbouring quasienergies approach a minimal nonzero distance when λ changes. In fact, as numerical experiments show, for systems which are classically chaotic such avoided crossings are abundant [20]. It was shown in [20] that the fictitious time elapsing between consecutive collisions scales as $N^{-\nu}$, $\nu > 0$.

Due to collisions the gas reaches state in which the motion of particles consists of fluctuations in the vicinity of equilibrium. Let us recall that in the original formulation of Pechukas i Yukawa it was assumed that H_0 is integrable and chaos develops gradually after switching the perturbation V and increasing the coupling strength λ, so the spectrum of $F(\lambda)$ conforms to RMT predictions only after certain 'relaxation (fictitious) time' when the phase space regions of regular motion have shrunk to relatively negligible weight. An ambitious program was thus designed to actually investigate the transition between spectra of integrable ($\lambda = 0$) and non-integrable (λ—large) cases. Thus, treating λ as a fictitious time, one faces a problem belonging to non-equilibrium rather than equilibrium statistical mechanics, and usefulness of tools of the latter could be doubtful. If, on the other hand, as argued in [20], H_0 and V are both non-integrable the initial state of the fictitious gas is already close to equilibrium. We are thus facing much less demanding tasks of finding the proper equilibrium distribution, knowing that the equilibrium is reached.

If the motion is ergodic (as it was shown [11] it is indeed ergodic although not on the whole energy surface), λ averages of spectral characteristics like the distribution of spacings between adjacent quasienergy levels equal ensemble averages. Strictly speaking, they are

equal if the fictitious time average is taken for $\lambda \to \infty$, but obviously they become practically undistinguishable after sufficiently many collisions, and it is enough to average over an interval $\Delta\lambda$ containing finite number (say M) of avoided crossings. Due to above-mentioned scaling of the fictitious time between collisions, in the semiclassical limit $N \to \infty$ the interval $\Delta\lambda$ shrinks to zero what substantiate applicability of the ensemble average predictions to a typical system at single value of λ.

The most straightforward application of statistical mechanics is to employ the canonical ensemble for the distribution of the dynamical variables (q, p, l),

$$\rho(q, p, l) \propto \exp(-\beta \mathcal{H}(q, p, l)). \tag{25}$$

A straightforward integration over Gauss-distributed p and l gives precisely the eigenphase density of random-matrix theory [2], i.e.

$$P(q_1, \ldots, q_N) = \int \mathrm{d}^N p \, \mathrm{d}^{N(N-1)/2} l \, \mathrm{e}^{-\beta \mathcal{H}} \propto \prod_{m<n} |\mathrm{e}^{-\mathrm{i}q_m} - \mathrm{e}^{-\mathrm{i}q_n}|. \tag{26}$$

As explained in the introduction the reasoning would be reasonable, if there were no other integrals of motion beside $\mathcal{H}$ itself. In the case other integrals of motion I_μ exist, the appropriate ensemble to use is the generalized canonical ensemble (12).

4. Integrals of motion

Equations (3), (4), and (5) are clearly integrable (they can be solved simply by diagonalizing F at given λ and calculating appropriate matrix elements), so one should expect that there are much more integrals of motion than the Hamilton function (7) itself. Indeed from (20) and (21) we see that the quantities

$$I_{k_1 m_1 \ldots k_n m_n} = \mathrm{Tr}(v^{k_1} l^{m_1} \cdots v^{k_n} l^{m_n}), \tag{27}$$

are indeed constants of motion, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} I_{k_1 m_1 \ldots k_n m_n} = 0, \tag{28}$$

and should be taken into account when constructing the generalized canonical ensemble (12). As already mentioned, such an ensemble yields the distribution of level spacings as well as low-order correlation functions of the level density in common with random-matrix theory, to within corrections of order $1/N$ [3, 12, 13]. The only problem is whether all integrals nailing down an invariant manifold are of the form (27).

It was shown that only $N^2 - N$ of such integrals are independent [14]. The independent integrals can be chosen in the form

$$C_k = \mathrm{Tr}\, v^k, \qquad k = 1, 2, \ldots, N-1, \tag{29}$$

$$C_{km} := \mathrm{Tr}(\mathrm{e}^{\mathrm{i}Q} v^k \mathrm{e}^{-\mathrm{i}Q} v^m), \qquad k, m = 1, 2, \ldots, N-1. \tag{30}$$

Using the definition of l (19) it is easy to show that the quantities (30) are linear combinations of (27).

Let us now count how many variables we have in (3)–(5). The variables q_n are real as eigenphases of the unitary matrix F and there are N of them. Also p_n as diagonal elements of a Hermitian matrix v are real, there are N of them as well. Since l is antihermitian and off-diagonal there are $(N^2 - N)/2$ matrix elements l_{mn}, but since they are, in the case of a general unitary matrix, complex, we should count separately their real and imaginary parts. Finally thus we have $N + N + (N^2 - N) = N^2 + N$ real variables. Comparing this with the

number of found integrals of motion $(N^2 - N)$ we are tempted to think that, in a generic case, invariant manifolds are of the dimension $(N^2 + N) - (N^2 - N) = 2N$.

On the other hand, in the coordinate frame in which V is diagonal, $V_{ij} = V_i \delta_{ij}$, the motion described by $F(\lambda) = \exp(-iV)U_0$ involves only N independent frequencies V_k, and takes place on an N-dimensional torus (and is ergodic on it in a generic case when the eigenvalues of V are not rationally dependent). It seems thus, we are still missing N independent integrals of motion.

5. Poisson structure and constraints

Before identifying missing integrals and determining their influence (or lack of) on the distribution of eigenphases, let us consider more carefully the proposed Hamiltonian formulation. First, observe that the definition of the manifold on which the level dynamics takes place as parameterized by the coordinates q_n, p_n and l_{mn} and equipped with the Poisson structure (8)–(10) is slightly flawed. From the definition (19) of l we have $l_{nn} = 0$, but this is inconsistent with the Jacobi identity which must be fulfilled by (10):

$$\{l_{pq}, \{l_{ik}, l_{mn}\}\} + \{l_{ik}, \{l_{mn}, l_{pq}\}\} + \{l_{mn}, \{l_{pq}, l_{ik}\}\} = 0. \tag{31}$$

Indeed, substituting to the above $m = k$ and $n = i$ and using $l_{kk} = l_{ii} = 0$ whenever they appear on intermediate steps, we arrive at an erroneous result $\delta_{pq} l_{kq} + \delta_{iq} l_{pi} - \delta_{kq} l_{pk} - \delta_{pi} l_{iq}$, instead of zero. Thus we are not allowed to put $l_{nn} = 0$ from the very beginning as equations defining our manifold. Instead, if we want to keep the Poisson brackets (10), we should change the definition (19) to

$$l = i\,e^{iQ}[v, e^{-iQ}] + iL, \tag{32}$$

where L is an arbitrary, real, diagonal matrix, i.e. we introduced N additional dynamical variables. To understand their meaning let us return to the derivation of the dynamical equations by diagonalizing matrix W (16), but this time we do not impose additional conditions on W, i.e. we do not assume that the diagonal matrix elements a_{nn} of $a = dW/d\lambda \cdot W^{-1}$ vanish. Instead we allow them to be arbitrary functions of λ. It should be clear (and indeed we will show that it is the case), nothing really depends on the choice of a_{nn}, since nothing concerning the eigenvalues should depend on the choice of the diagonalizing matrix.

The resulting equations of motion are derived in the same way as previous ones (3)–(5). In fact only the third of them is altered and reads now

$$\frac{dl_{mn}}{d\lambda} = - \sum_{k \neq m,n} l_{mk} l_{kn} (\mathcal{V}(q_n - q_k) - \mathcal{V}(q_k - q_m))$$
$$+ l_{mn}(a_{mm} - a_{nn}) + l_{mn}(l_{nn} - l_{mm})\mathcal{V}(q_m - q_n). \tag{33}$$

Equations (3), (4), (33) are again Hamiltonian with the same Poisson structure (8)–(10), but with a new Hamilton function

$$\mathcal{H} = \frac{1}{2} \sum_{n=1}^{N} p_n^2 + \frac{1}{2} \sum_{n,m=1}^{N} l_{mn} l_{nm} \mathcal{V}(q_n - q_m) + \sum_{j}^{N} a_{jj} l_{jj}, \tag{34}$$

depending on N arbitrary (in general fictitious time-, i.e. λ-dependent) functions a_{nn}. The quantities C_{mn} (30) are again integrals of motion. In addition, we easily calculate that

$$\{\mathcal{H}, l_{nn}\} = 0, \tag{35}$$

so l_{nn} are also constants of motion. In fact, as it is clear from the previous considerations, nothing concerning the eigenphases depends on actual values of l_{nn}. We can thus impose constraints, e.g.

$$l_{nn} = 0, \qquad n = 1, \dots, N. \tag{36}$$

At this point it is instructive and in fact very natural to describe the encountered situation from the point of view of Dirac's theory of constrained Hamiltonian systems [21]. Conditions (36) are so called primary constraints (i.e. they are not obtained from the equations of motion) and can be imposed only *after* evaluating all Poisson brackets to avoid the problems with the Jacobi identity mentioned at the beginning of the present section. Further, one calculates easily that

$$\{l_{mm}, l_{nn}\} = 0. \tag{37}$$

Together with (35) it means that the consistency condition

$$\{\mathcal{H}, l_{mm}\} + \sum_{n=1}^{N} a_{nn}\{l_{nn}, l_{mm}\} = 0 \tag{38}$$

is identically fulfilled and no other constraints, neither primary nor secondary, are produced, nor additional conditions are imposed on the functions $a_{nn}(\lambda)$.

Due to (37) l_{nn} are automatically first-class constraints (recall that according to Dirac terminology a quantity is of first class if its Poisson brackets with all constraints vanish, see [21], p 18). The new Hamilton function (34) involves as many arbitrary functions (in our case these are functions $a_{nn}(\lambda)$), as there are independent primary first-class constraints. On the other hand, first-class primary constraints (36) may be always used to produce a gauge transformation generated by

$$G(\lambda) = \sum_{n=1}^{N} \theta_n(\lambda) l_{nn}, \tag{39}$$

i.e.

$$l_{ij} \mapsto e^{i\theta_i(\lambda)} l_{ij} \, e^{-i\theta_j(\lambda)}, \tag{40}$$

with arbitrary λ-dependent $\theta_k, k = 1, \dots, N$.

We expect that an initial physical state determined by initial values of the phase-space variables (q_n, p_n, l_{mn}) determines also its all future physical states. Since the Hamilton function (34) depends on N arbitrary functions, the same may happen to the values of (q_n, p_n, l_{mn}) at later (fictitious) times. But the only freedom is now given by the gauge transformation (40) connecting the variables describing the same physical state of the system for different choices of the gauge. Hence particular physical properties of the state (e.g. statistical properties of the distribution of positions, i.e., in our case, eigenphases) should be gauge independent, and in fact they are, since the gauge transformation does not influence the relevant variables q_n.

To be even more concrete in explaining the role of the gauge transformation for the present problem let us observe that by assuming $l_{mm} = 0$ we recovered the previous count of the number of variables versus dimension of the invariant manifold, since the number of variables was first increased by N by introducing the diagonal elements of l and then decreased by the same number by imposing constraints equating them to zero. To fix the (still) remaining N degrees of freedom let us observe that the gauge transformation (40) does not change the integrals of motion (in particular the Hamilton function itself) after reducing to the manifold determined by the constraints (36), retaining also the equations of motion in their original

form. The transformation is intimately related to the freedom of choice of the diagonalizing matrix W in terms of a, it leads to

$$a \mapsto i\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} + \mathrm{e}^{\mathrm{i}\theta} a\, \mathrm{e}^{-\mathrm{i}\theta}, \qquad \theta := \mathrm{diag}(\theta_1, \ldots, \theta_2). \tag{41}$$

With the help of (40) we can fix in an arbitrary way N (more precisely $N-1$, but one additional is determined by the choice of initial point on the unit circle) phases of the variables l_{nm}. Let us summarize

- number of variables: $N_{\mathrm{var}} = N^2 + 2N$ (the old ones plus the (imaginary parts of) diagonal elements of l);
- number of independent integrals C_{mn}: $N_{\mathrm{int}} = N^2 - N$;
- number of constraints $l_{nn} = 0$: $N_c = N$;
- number of phases fixed by choosing a gauge $N_g = N$;

hence, $N_{\mathrm{var}} - (N_{\mathrm{int}} + N_c + N_g) = N = $ dimension of the invariant manifold.

Now it is clear that integrals of motion C_{mn} (30) are the only quantities which should be taken into account when determining the equilibrium distribution. Indeed, as already mentioned the constraints and the gauge, involving only l_{mn}, do not influence eigenphases, which is a direct consequence of the independence of the eigenvalues on the choice of the diagonalizing matrix. Moreover, our choice $l_{nn} = 0$ reduces the Hamilton function (34) to originally considered one (7) and the whole reasoning which led, after integration out of p and l variables and neglecting corrections of order $1/N$, to random matrix results for the eigenphases, is fully vindicated.

Acknowledgments

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